CALT-TH-2020-018

The F-Symbols for Transparent Haagerup-Izumi Categories with $G = \mathbb{Z}_{2n+1}$

Tzu-Chen Huang, Ying-Hsuan Lin

Walter Burke Institute for Theoretical Physics,
California Institute of Technology, Pasadena, CA 91125, USA

jimmy@caltech.edu, yhlin@caltech.edu

Abstract

The notion of a transparent fusion category is defined. For the Haagerup-Izumi fusion rings with $G = \mathbb{Z}_{2n+1}$ (the $\mathbb{Z}_3$ case is the Haagerup $\mathcal{H}_3$ fusion ring), the transparent property reduces the number of independent F-symbols from order $\mathcal{O}(n^6)$ to $\mathcal{O}(n^2)$, rendering the pentagon identity practically solvable. Transparent Haagerup-Izumi categories are constructed up to $\mathbb{Z}_{15}$, and the explicit F-symbols are compactly presented. The potential construction of categories for new families of fusion rings is discussed.

Contents

1 Introduction

Subfactors [1,2] and fusion categories [3,4] provide the mathematical framework underlying various physical objects in quantum field theory, including anyons in (2+1)d Chern-Simons theory/(1+1)d rational conformal field theory [5–8] and topological defect lines in (1+1)d quantum field theory [9–11]. Fusion categories with non-invertible objects generalize the notion of symmetries and 't Hooft anomalies in quantum field theory [11–14]. Due to Ocneanu rigidity [15,3], a category is an invariant under renormalization group flows connecting short and long distance physics. This generalization of the 't Hooft anomaly matching condition has shed new light on the phases of quantum field theory.

Subfactor theory has an inherent categorical structure [16], and has been a productive factory of fusion categories.[1] Subfactors with Jones indices less than 4 have been classified by Ocneanu [17], and extended to 4 by Popa [18]. Haagerup [19] then searched for subfactors with Jones indices *a little bit beyond 4*, and together with Asaeda [20] constructed one with Jones index $\frac{5+\sqrt{13}}{2}$, the smallest above 4. In [21], Izumi generalized the Haagerup fusion ring to a family of fusion rings labeled by the finite abelian group G of invertible objects, and explicitly constructed the subfactors for $G = \mathbb{Z}_3$, $\mathbb{Z}_5$. The constructive classification of subfactors for $|G|$ odd was achieved up to $|G| = 19$ by Evans and Gannon [22], and of subfactors with $G = \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_6$, $\mathbb{Z}_8$, $\mathbb{Z}_{10}$ by Grossman, Izumi, and Snyder [23–26].

The key data underlying a fusion category are the F-symbols, which are solutions to the pentagon identity. Some (almost) equivalent notions exist: associators, quantum $6j$-symbols and crossing kernels. They underlie Turaev-Viro theory [27, 28] and Levin-Wen string-net models [29]. In [11], one of the present authors showed how the F-symbols strongly constrain (1+1)d (fully extended) topological quantum field theory [30, 31]; in many cases, the F-symbols completely determine the full field theory data by bootstrap.

In this paper, unitary and non-unitary Haagerup-Izumi categories with $G = \mathbb{Z}_{2n+1}$ are

[1] At the level of the fusion ring, the bimodules of a subfactor correspond to the simple objects in a fusion category, with the tensor product inherited.

constructed up to $\mathbb{Z}_{15}$ by directly solving the pentagon identity. We first define the utilitarian notion of a *transparent* fusion category (see Definition 3.1), and derive various graph equivalences and F-symbol relations that reduce the number of independent F-symbols from $\mathcal{O}(n^6)$ to $\mathcal{O}(n^2)$, rendering the pentagon identity practically solvable. Of course, there may exist non-transparent fusion categories that elude this approach. These relations are summarized into a T-system of constraints (see Definition 4.1), and the solutions to the pentagon identity under the T-system constraints provide a classification of F-symbols for transparent Haagerup-Izumi categories. The result of this classification is stated in Theorems 5.1 and 5.2.

It should be stressed that whenever a subfactor construction exists, the F-symbols of the corresponding fusion category are completely fixed by the subfactor data [32]. Thus, the F-symbols for various unitary Haagerup-Izumi categories were in principle already determined by Izumi [21], Evans and Gannon [22] using Cuntz algebra techniques. Evans and Gannon further generalized such constructions to fusion categories that need not have subfactor realizations and need not be unitary [33]. Moreover, the explicit F-symbols for the Haagerup $\mathcal{H}_3$ ($G = \mathbb{Z}_3$) fusion category have been recently computed using the pentagon approach by [34,35]. The purpose of this paper is firstly, to offer the pentagon construction for Haagerup-Izumi categories, and secondly, to point out the existence of a particularly simple gauge with the transparent property that all F-symbols involving at least one external invertible object are all 1.[2] In physical applications, such a gauge allows the effective exploitation of the $\mathbb{Z}_{2n+1}$ symmetry.

The outline of this paper is as follows. Section 2 reviews the string diagram calculus for pivotal fusion categories. Section 3 introduces the notion of a transparent fusion category, and derives various consequences including invariance relations for the F-symbols. Section 4 introduces the Haagerup-Izumi fusion rings, and formulates a T-system of constraints on the F-symbols that must be satisfied for transparent fusion categories. Section 5 presents the classification of solutions to the pentagon identity under the T-system constraints, and the explicit F-symbols for transparent Haagerup-Izumi categories. Section 6 ends with some concluding remarks.

Note: The authors first obtained the F-symbols for the Haagerup $\mathcal{H}_3$ fusion category from Matthew Titsworth [34]. By performing gauge transformations on his solution, a transparent gauge was found. This observation led the present authors to postulate that transparent fusion categories also exist for the subsequent Haagerup-Izumi fusion rings.

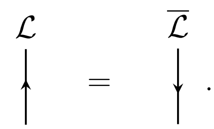
[2] In [34, 35], the F-symbols for the Haagerup $\mathcal{H}_3$ ($G = \mathbb{Z}_3$) fusion category were presented in non-transparent gauges.

2 Preliminary

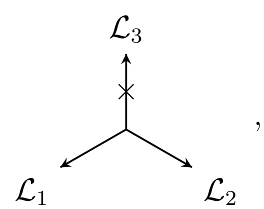
A classic introduction to fusion categories can be found in [3,4]. The type of fusion categories considered in this paper are pivotal fusion categories in which every object $\mathcal{L}$ has a two-sided dual $\overline{\mathcal{L}}$.[3]

2.1 String diagrams and gauge freedom

In this paper, the notation for string diagrams is as follows. Each object $\mathcal{L}$ is represented by an oriented string, which is equivalent to its dual $\overline{\mathcal{L}}$ with the opposite orientation,



The basic building block for string diagrams is a trivalent junction composed of a vertex and three open edges



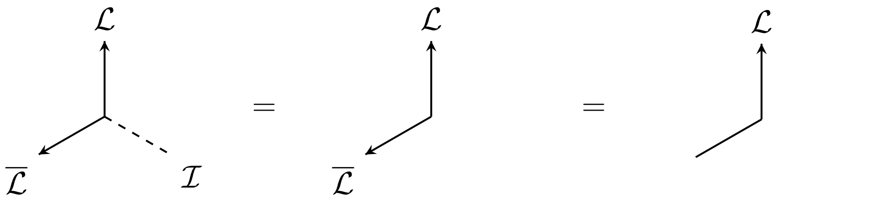
where $\times$ specifies the ordering of the edges. This trivalent junction represents the morphism

$$V_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3} \equiv \hom(\overline{\mathcal{L}}_2 \otimes \overline{\mathcal{L}}_1, \mathcal{L}_3) \in \mathbb{C}^{N_{\mathcal{L}_3}^{\overline{\mathcal{L}}_2,\overline{\mathcal{L}}_1}} ,$$

where $N_{\mathcal{L}_3}^{\overline{\mathcal{L}}_2,\overline{\mathcal{L}}_1}$ is the fusion coefficient, *i.e.* the multiplicity of $\mathcal{L}_3$ in $\overline{\mathcal{L}}_2 \otimes \overline{\mathcal{L}}_1$. A change of basis at this vertex is a gauge transformation $g_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3} \in GL(N_{\mathcal{L}_3}^{\mathcal{L}_1,\mathcal{L}_2}, \mathbb{C})$. To simplify the discussion, it is assumed in the following that all nonzero fusion coefficients are one, so that every $g_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}$ is a complex scalar. Finally, edges representing the unit object $\mathcal{I}$ can be

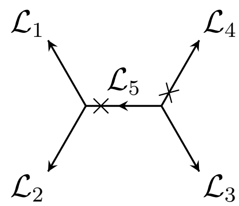
[3]For such categories, a physical formulation in the context of topological defect lines in (1+1)d quantum field theory can be found in [11] (see also [9, 10]). In this context, string diagrams — which were first introduced in [36] to facilitate calculations and proofs — are in fact the physical objects.

removed[4]

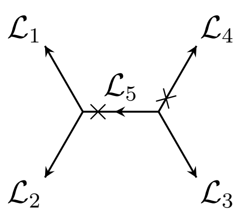


2.2 F-move and pentagon identity

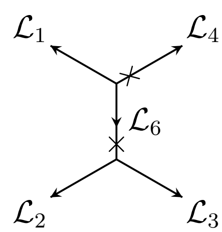
For a string diagram composed of two trivalent junctions



the gauge freedom is $g_{\mathcal{L}_1,\mathcal{L}_2,\overline{\mathcal{L}}_5}\, g_{\mathcal{L}_5,\mathcal{L}_3,\mathcal{L}_4}$. It is related by an F-move to string diagrams of a different configuration,



$$= \sum_{\mathcal{L}_6} (F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\mathcal{L}_6}$$



,

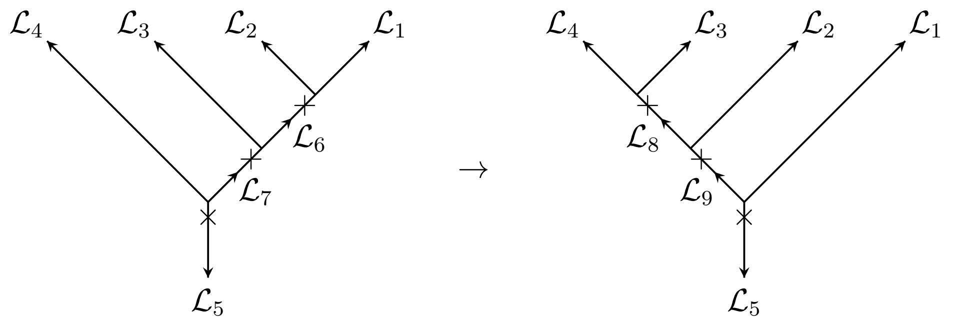
where $(F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\mathcal{L}_6}$ are the F-symbols. The gauge factor for an F-symbol is

$$\frac{g_{\mathcal{L}_1,\mathcal{L}_2,\overline{\mathcal{L}}_5}\, g_{\mathcal{L}_5,\mathcal{L}_3,\mathcal{L}_4}}{g_{\mathcal{L}_2,\mathcal{L}_3,\overline{\mathcal{L}}_6}\, g_{\mathcal{L}_1,\mathcal{L}_6,\mathcal{L}_4}} \,.$$

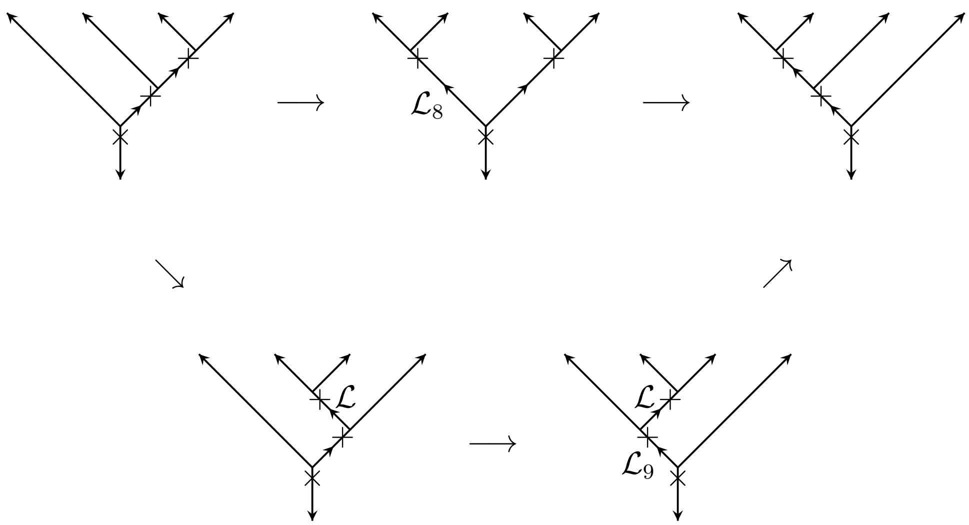
The F-symbols must satisfy a consistency condition that is the equivalence of two different

[4]There is no ordering/marking at a trivalent junction when one of the objects is the unit.

combinations of F-moves that result in



To be precise, the equivalence of the two routes



gives the pentagon identity

$$(F^{\mathcal{L}_6,\mathcal{L}_3,\mathcal{L}_4}_{\overline{\mathcal{L}}_5})_{\mathcal{L}_7,\mathcal{L}_8}\,(F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_8}_{\overline{\mathcal{L}}_5})_{\mathcal{L}_6,\mathcal{L}_9} \;=\; \sum_{\mathcal{L}} (F^{\mathcal{L}_1,\mathcal{L}_2\mathcal{L}_3}_{\mathcal{L}_7})_{\mathcal{L}_6,\mathcal{L}}\,(F^{\mathcal{L}_1,\mathcal{L},\mathcal{L}_4}_{\overline{\mathcal{L}}_5})_{\mathcal{L}_7,\mathcal{L}_9}\,(F^{\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_4}_{\mathcal{L}_9})_{\mathcal{L},\mathcal{L}_8}\,. \tag{2.1}$$

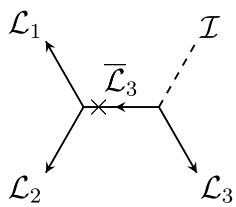
A solution to the pentagon identity amounts to a construction of a pivotal fusion category. If there are n isomorphism classes of simple objects, then the pentagon identity is a set of $\mathcal{O}(n^9)$ cubic polynomial equations for $\mathcal{O}(n^6)$ variables, modulo $\mathcal{O}(n^3)$ gauge freedom. As n grows, a generic system of this size quickly becomes impossible to solve.

2.3 Cyclic permutation map

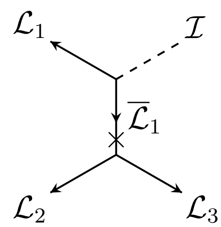
The cyclic permutation map is the isomorphism of the three vector spaces

$$V_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}\,, \quad V_{\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_1}\,, \quad V_{\mathcal{L}_3,\mathcal{L}_1,\mathcal{L}_2}\,,$$

which pictorially corresponds to moving the $\times$ mark around. It is the F-move with an external edge representing the unit object $\mathcal{I}$:



$$= \quad (F_{\mathcal{I}}^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3})_{\overline{\mathcal{L}}_3,\overline{\mathcal{L}}_1}$$



.

The net effect is a counter-clockwise rotation of the $\times$ mark and a factor of $(F_{\mathcal{I}}^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3})_{\overline{\mathcal{L}}_3,\overline{\mathcal{L}}_1}$.

It would be nice if the ordering and marking at the trivalent junctions could be forgotten, which requires the F-symbols $(F_{\mathcal{I}}^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3})_{\overline{\mathcal{L}}_3,\overline{\mathcal{L}}_1}$ to have value 1 for all $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$. As explained in Appendix A of [11], in any fusion category, the gauge freedom

$$g_{\mathcal{I},\mathcal{L},\overline{\mathcal{L}}}\,, \quad g_{\mathcal{L},\mathcal{I},\overline{\mathcal{L}}}\,, \quad g_{\mathcal{L},\overline{\mathcal{L}},\mathcal{I}} \qquad \forall\,\mathcal{L}$$

together with the pentagon identity could set

$$(F_{\mathcal{I}}^{\mathcal{I},\mathcal{L},\overline{\mathcal{L}}})_{\mathcal{L},\mathcal{I}} = (F_{\overline{\mathcal{L}}}^{\overline{\mathcal{L}},\mathcal{I},\mathcal{I}})_{\overline{\mathcal{L}},\mathcal{I}} = (F_{\mathcal{L}}^{\mathcal{I},\mathcal{I},\mathcal{L}})_{\mathcal{I},\mathcal{L}} = (F_{\mathcal{I}}^{\mathcal{L},\overline{\mathcal{L}},\mathcal{I}})_{\mathcal{I},\overline{\mathcal{L}}} = (F_{\mathcal{I}}^{\overline{\mathcal{L}},\mathcal{I},\mathcal{L}})_{\overline{\mathcal{L}},\mathcal{L}} = (F_{\mathcal{L}}^{\mathcal{I},\mathcal{L},\mathcal{I}})_{\mathcal{L},\mathcal{L}} = 1 \quad \forall\,\mathcal{L}\,,$$

and

$$(F_{\overline{\mathcal{L}}_3}^{\mathcal{I},\mathcal{L}_1,\mathcal{L}_2})_{\mathcal{L}_1,\overline{\mathcal{L}}_3} = (F_{\overline{\mathcal{L}}_1}^{\mathcal{L}_2,\mathcal{L}_3,\mathcal{I}})_{\overline{\mathcal{L}}_1,\mathcal{L}_3} = (F_{\mathcal{I}}^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3})_{\overline{\mathcal{L}}_3,\overline{\mathcal{L}}_1} = 1 \qquad \forall\,\mathcal{L}_1,\,\mathcal{L}_2,\,\mathcal{L}_3\,,$$

but is insufficient to set

$$(F_{\mathcal{I}}^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3})_{\overline{\mathcal{L}}_3,\overline{\mathcal{L}}_1} = 1 \qquad \forall\,\mathcal{L}_1,\,\mathcal{L}_2,\,\mathcal{L}_3\,.$$

Definition 2.1 (Cyclic) *A pivotal fusion category is cyclic if there exists a gauge in which all F-symbols with at least one external unit object have value 1. Such a gauge is called a cyclic gauge. A gauge transformation g is cyclic if*

$$g_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3} = g_{\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_1} = g_{\mathcal{L}_3,\mathcal{L}_1,\mathcal{L}_2} \qquad \forall\,\mathcal{L}_1,\,\mathcal{L}_2,\,\mathcal{L}_3\,.$$

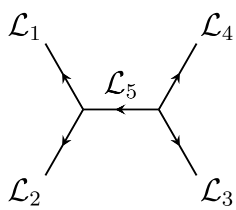
Given a cyclic fusion category in a cyclic gauge, if the gauge transformations are further restricted to be cyclic, then the ordering and marking on the trivalent junctions can be fully ignored.

2.4 Tetrahedral symmetry

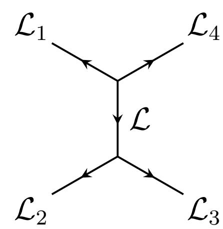
Consider a cyclic fusion category $\mathcal{C}$ in a cyclic gauge. It is clear by a π rotation that the F-symbols enjoy a $\mathbb{Z}_2$ invariance relation

$$(F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\mathcal{L}_6} = (F^{\mathcal{L}_3,\mathcal{L}_4,\mathcal{L}_1}_{\overline{\mathcal{L}}_2})_{\overline{\mathcal{L}}_5,\overline{\mathcal{L}}_6} \,.$$

Additional relations can be derived as follows. On both sides of the F-move equation

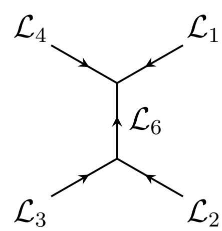


$$= \quad \sum_{\mathcal{L}} (F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\mathcal{L}}$$



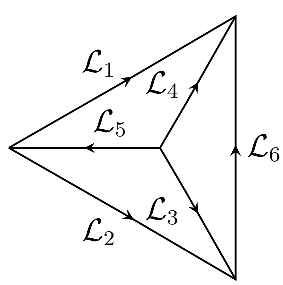
,

join



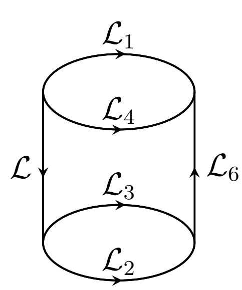
(2.2)

from the right. The resulting graph on the left side of the F-move equation can be adjusted into a tetrahedron

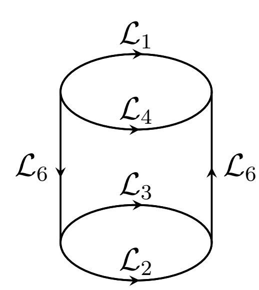


,

whereas the graph on the right side of the F-move equation can be adjusted into

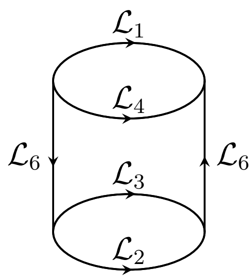


$$= \delta_{\mathcal{L},\mathcal{L}_6} \times$$

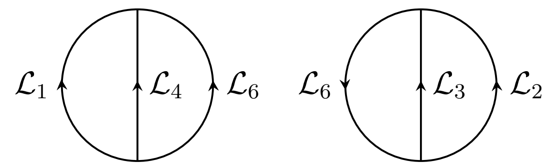


,

which vanishes if $\mathcal{L} \neq \mathcal{L}_6$ because the top and bottom loops can be shrunk but the vector space $V_{\mathcal{L},\mathcal{L}_6}$ is empty. Applying the F-move to a unit object connecting the two $\mathcal{L}_6$ edges gives



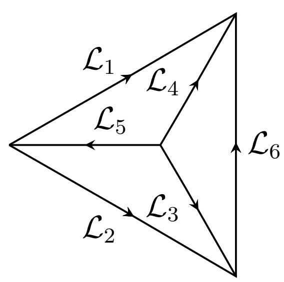
$$= \quad (F^{\overline{\mathcal{L}}_6,\mathcal{L}_6,\overline{\mathcal{L}}_6}_{\overline{\mathcal{L}}_6})_{\mathcal{I},\mathcal{I}}$$



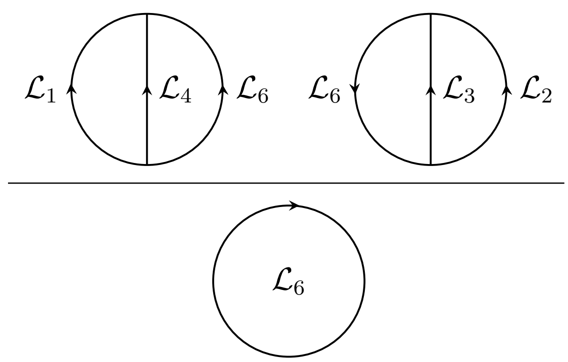
.

Again, no non-unit object $\mathcal{L}$ can bridge the two Θ graphs on the right because the Θ graphs can be shrunk, but the vector space $V_{\mathcal{L},\mathcal{I}}$ is empty if $\mathcal{L} \neq \mathcal{I}$.

Putting things together,

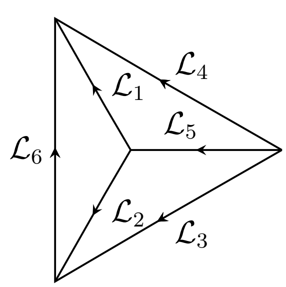


$$= \quad (F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\mathcal{L}_6}$$

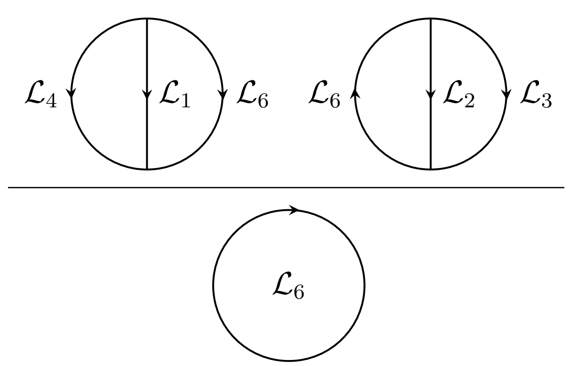


. (2.3)

A similar derivation by joining (2.2) from the left with the F-move equation shows that



$$= \quad (F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\mathcal{L}_6}$$



. (2.4)

The two tetrahedra in (2.3) and (2.4) are related by a spherical move, and are equal if $\mathcal{C}$ is spherical [37]. Each tetrahedron enjoys an S_3 symmetry: it is invariant under the $\mathbb{Z}_3$ rotations and complex conjugate under a reflection. Combined with the spherical move, there is an S_4 worth of relations for the F-symbols.

3 Transparent fusion categories

Definition 3.1 (Transparent) *A pivotal fusion category is transparent if there exists a gauge in which all F-symbols with at least one external invertible object have value 1. Such a gauge is called a transparent gauge.*

Corollary 3.1 *Every transparent fusion category is cyclic, and every transparent gauge is a cyclic gauge.*

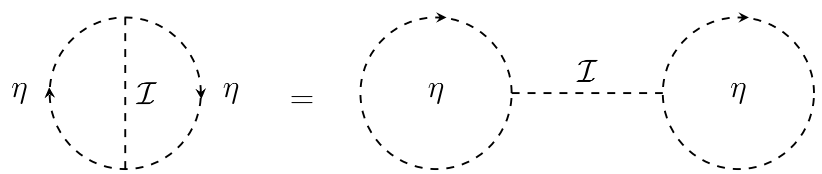
Corollary 3.2 *In a transparent fusion category, the group G formed by the invertible objects is non-anomalous, i.e. the F-symbols belong to the trivial cohomology class in $H^3(G, U(1))$.*

In the following, properties of transparent fusion categories are derived, including graph equivalences and F-symbol invariance relations. By Corollary 3.1, in a transparent gauge, the ordering and marking on the trivalent junctions are ignored.

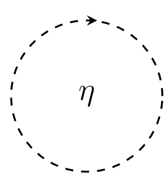
3.1 Graph equivalences

Let $\mathcal{C}$ be a transparent fusion category in a transparent gauge, and η an invertible object. There are the following graph equivalences.

1. **(Loop Value)** Applying the F-move to an invertible η loop gives



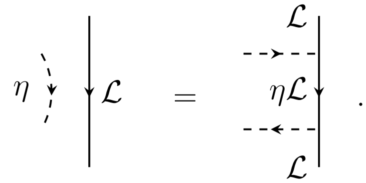
where the transparent property is used to set $(F^{\bar{\eta},\eta,\bar{\eta}}_{\bar{\eta}})_{\mathcal{I},\mathcal{I}} = 1$. Thus



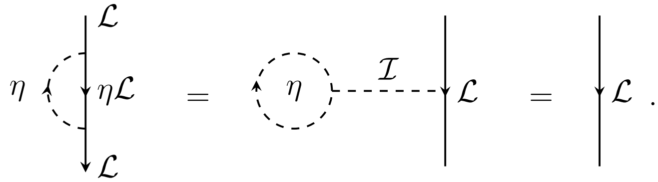
$$= 1\,, \tag{3.1}$$

i.e. invertible loops have value 1.

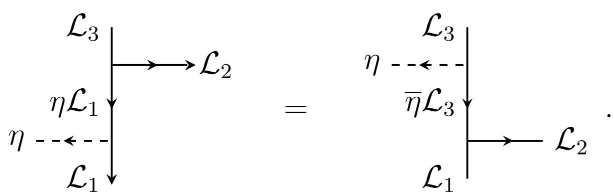
2. **(Attachment)** An invertible object can be attached to a simple object $\mathcal{L}$



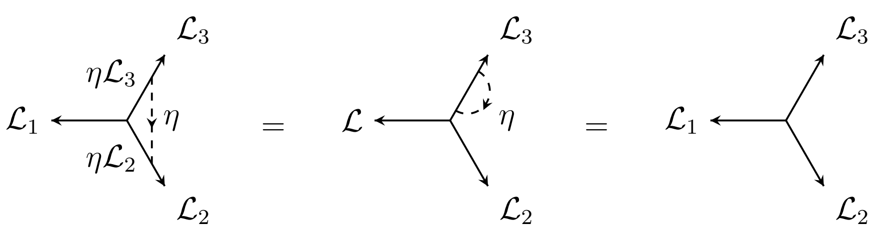
3. **(Detachment)** An invertible object with two ends attached to a non-invertible simple object $\mathcal{L}$ can be detached



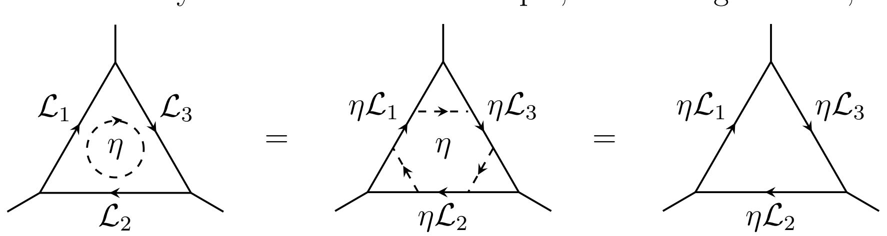
4. **(Swap)** An invertible object attached to an edge can be swapped across a trivalent junction



5. **(Contraction)** An invertible object bridged across a trivalent junction can be contracted. It can be regarded as a swap followed by a detachment

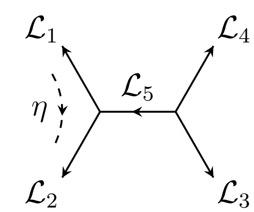


6. **(Symmetry nucleation)** Given a graph, an invertible loop can be nucleated on any face and merged with the bordering edges, where the merging can be regarded as attachments followed by contractions. For example, on a triangular face,

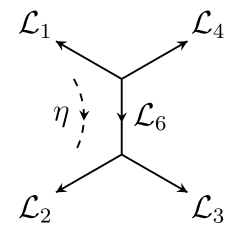


3.2 F-symbol invariance relations

A slight variation of symmetry nucleation results in invariance relations for F-symbols. An invertible object η is first added to an open face

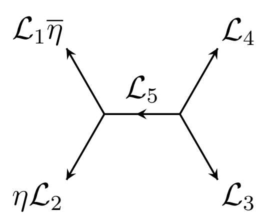


$$= \quad \sum_{\mathcal{L}} (F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\mathcal{L}_6}$$

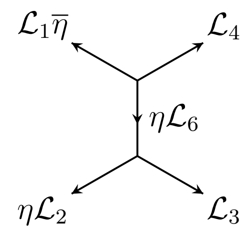


,

and then attached and contracted



$$= \quad \sum_{\mathcal{L}} (F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\mathcal{L}_6}$$



.

The result is an invariance relation

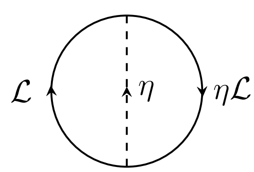
$$(F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\mathcal{L}_6} \;=\; (F^{\mathcal{L}_1\overline{\eta},\eta\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\eta\mathcal{L}_6}\,.$$

Symmetry nucleation on the other three faces gives

$$(F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\mathcal{L}_6} \;=\; (F^{\mathcal{L}_1,\mathcal{L}_2\overline{\eta},\eta\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5\overline{\eta},\mathcal{L}_6} \;=\; (F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3\overline{\eta}}_{\eta\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\mathcal{L}_6\overline{\eta}} \;=\; (F^{\eta\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}_4\overline{\eta}}})_{\eta\mathcal{L}_5,\mathcal{L}_6}\,.$$

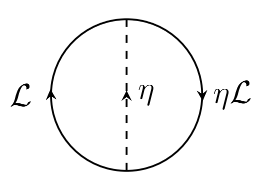
3.3 Θ graphs and tetrahedra

Transparency implies relations for Θ graphs and tetrahedra. Let η be an invertible object. Consider the Θ graph

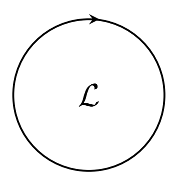


.

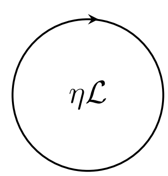
By detachments,



$=$

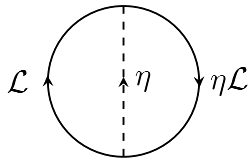


$=$

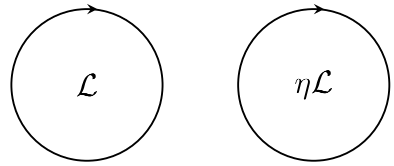


.

Apply the F-move to η to obtain

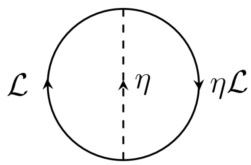


$$= \quad (F^{\eta\mathcal{L},\mathcal{L},\mathcal{L}}_{\overline{\eta\mathcal{L}}})_{\eta,\mathcal{I}}$$

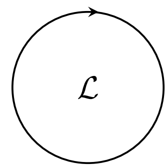


,

where no non-unit object $\mathcal{L}'$ can bridge the two loops on the right because the loops can be shrunk and the junction vector space $V_{\mathcal{L}',\mathcal{I}}$ is empty if $\mathcal{L}' \neq \mathcal{I}$. From the above, it can be deduced that

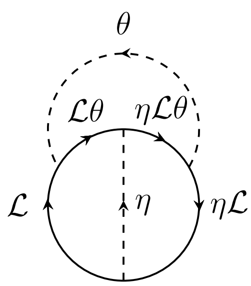


$$= \quad (F^{\eta\mathcal{L},\overline{\mathcal{L}},\mathcal{L}}_{\eta\mathcal{L}})_{\eta,\mathcal{I}}{}^{-1} \quad =$$

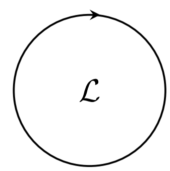


.

Next consider tetrahedra. Let η, θ be invertible lines. Then

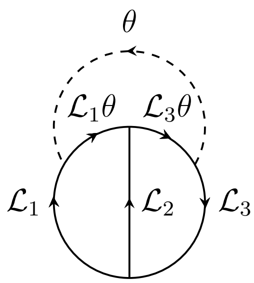


$$= \quad (F^{\eta\mathcal{L}\theta,\overline{\mathcal{L}\theta},\mathcal{L}}_{\eta\mathcal{L}})_{\eta,\overline{\theta}}{}^{-1} \quad = \tag{3.2}$$

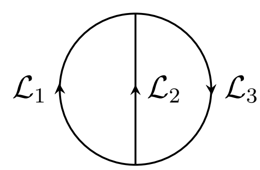


,

where the first equality is obtained by applying an F-move to η, and the two graphs are equivalent by a swap and two detachments. More generally,

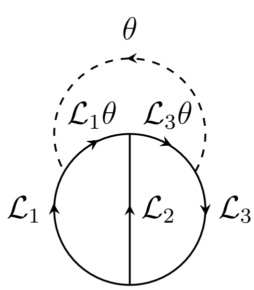


$$= \tag{3.3}$$

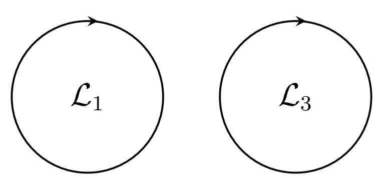


.

Apply an F-move to $\mathcal{L}_2$ to obtain

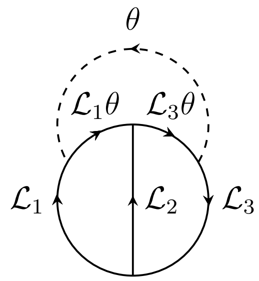


$$= \quad (F^{\mathcal{L}_1,\overline{\mathcal{L}}_3,\mathcal{L}_3\theta}_{\mathcal{L}_1\theta})_{\overline{\mathcal{L}}_2,\theta} \tag{3.4}$$

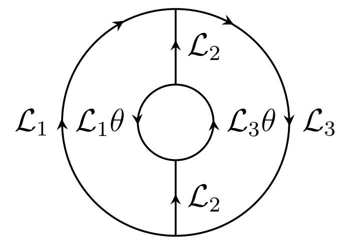


.

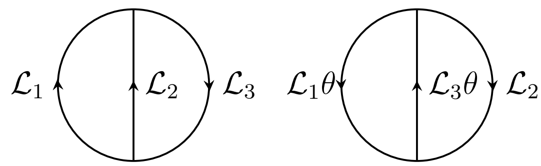
Independently, apply an F-move to θ and then to a unit object connecting the two $\mathcal{L}_2$ edges to obtain



$$= \quad (F^{\overline{\mathcal{L}}_1,\mathcal{L}_1\theta,\overline{\mathcal{L}_3\theta}}_{\overline{\mathcal{L}}_3})_{\theta,\overline{\mathcal{L}}_2}$$



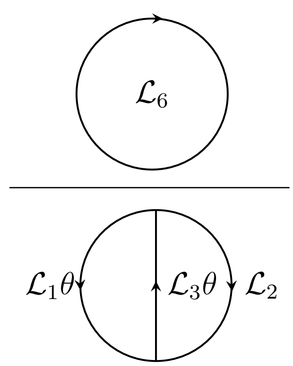
$$= \quad (F^{\overline{\mathcal{L}}_1,\mathcal{L}_1\theta,\overline{\mathcal{L}_3\theta}}_{\overline{\mathcal{L}}_3})_{\theta,\overline{\mathcal{L}}_2} \quad (F^{\overline{\mathcal{L}}_1,\mathcal{L}_1\theta,\overline{\mathcal{L}_3\theta}}_{\overline{\mathcal{L}}_3})_{\mathcal{I},\mathcal{I}}$$



.

Combined with (3.2) and (3.3),

$$(F^{\overline{\mathcal{L}}_1,\mathcal{L}_1\theta,\overline{\mathcal{L}_3\theta}}_{\overline{\mathcal{L}}_3})_{\theta,\overline{\mathcal{L}}_2} \quad = \tag{3.5}$$



.

4 Transparent Haagerup-Izumi categories

A Haagerup-Izumi fusion ring can be defined for any finite abelian group G. A key feature is that it is quadratic: the fusion of a single non-invertible simple object with the invertible objects generate all the non-invertible simple objects. In this section, properties of transparent fusion categories are applied to the Haagerup-Izumi fusion rings with $G = \mathbb{Z}_{2n+1}$ to formulate a T-system of constraints for classifying transparent Haagerup-Izumi categories.

4.1 The Haagerup-Izumi fusion rings

The Haagerup-Izumi category with $G = \mathbb{Z}_\nu$ has ν invertible objects

$$\mathcal{I}, \quad \alpha, \quad \alpha^2, \quad \cdots \quad \alpha^{\nu-1}$$

and ν non-invertible simple objects

$$\rho, \quad \alpha\rho, \quad \alpha^2\rho, \quad \cdots \quad \alpha^{\nu-1}\rho\,.$$

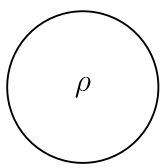
The fusion ring is

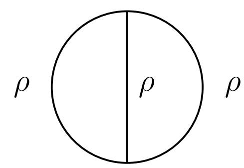
$$\alpha^\nu = 1\,, \quad \alpha\rho = \rho\,\alpha^{\nu-1}\,, \quad \rho^2 = \mathcal{I} + \sum_{k=0}^{\nu-1} \alpha^k \rho\,.$$

When $\nu = 1$, this is the Fibonacci ring, which is the Grothendieck ring of the Fibonacci category (even sectors of the A_4 subfactor) and Lee-Yang category. When $\nu = 2$, this is the Grothendieck ring of the $\mathcal{C}(sl(2), 8)_{ad}$ fusion category (even sectors of the A_7 subfactor), which is premodular but not modular [38]. When $\nu = 3$, this is the Grothendieck ring of the Haagerup $\mathcal{H}_3$ fusion category. For $\nu \geq 3$, there is no braiding since the fusion ring is non-commutative.

4.2 Loops and Θ graphs

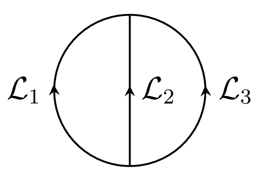
Let $\mathcal{C}$ be a transparent Haagerup-Izumi category with $G = \mathbb{Z}_{2n+1}$, and fix a transparent gauge. Define ζ and ξ to be the graph values

$\zeta \equiv$ 

, $\xi \equiv$ 

.

On the left, symmetry nucleation implies that all non-invertible loops have value ζ. On the right, symmetry nucleation on the three faces implies that all Θ graphs with three non-invertible simple objects have the same value ξ. Together with (3.1), these graph values can be summarized as



$$= \begin{cases} 1 & \text{all invertible}\,, \\ \zeta & \text{one invertible}\,, \\ \xi & \text{all non-invertible}\,. \end{cases}$$

By (3.2), for any invertible η and θ,

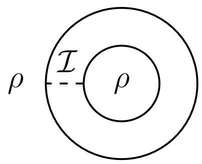
$$(F^{\eta\mathcal{L}\theta,\overline{\mathcal{L}\theta},\mathcal{L}}_{\eta\mathcal{L}})_{\eta,\bar{\theta}} = \begin{cases} 1 & \mathcal{L} \ \text{invertible}\,, \\ \zeta^{-1} & \mathcal{L} \ \text{non-invertible}\,. \end{cases}$$

The F-symbols with a single internal invertible object can also be deduced. Let $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ be non-invertible. By (3.4) and (3.5),

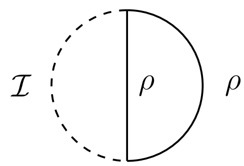
$$(F^{\mathcal{L}_1,\mathcal{L}_3,\eta\mathcal{L}_3}_{\mathcal{L}_1\overline{\eta}})_{\mathcal{L}_2,\overline{\eta}} \;=\; \zeta^{-2}\,\xi\,, \qquad (F^{\mathcal{L}_1,\eta\mathcal{L}_1,\eta\mathcal{L}_3}_{\mathcal{L}_3})_{\overline{\eta},\mathcal{L}_2} \;=\; \zeta\,\xi^{-1}\,.$$

From this, the possible values of ζ are constrained as follows. Consider two concentric ρ loops and apply the F-move to obtain[5]

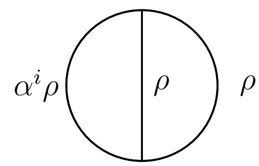
$$\zeta^2 \;=$$



$$=\; (F^{\rho,\rho,\rho}_{\rho})_{\mathcal{I},\mathcal{I}}$$



$$+\; \sum_{i=0}^{2n} (F^{\rho,\rho,\rho}_{\rho})_{\mathcal{I},\alpha^i\rho}$$



$$=\; 1+(2n+1)\,\zeta\,.$$

Hence,

$$\zeta \;=\; \frac{2n+1\pm\sqrt{(2n+1)^2+4}}{2}\,.$$

Since ξ is not gauge-invariant, a gauge can be chosen such that

$$\xi \;=\; \zeta^2\,, \qquad (F^{\mathcal{L}_1,\mathcal{L}_3,\eta\mathcal{L}_3}_{\overline{\eta\mathcal{L}_1}})_{\mathcal{L}_2,\overline{\eta}} \;=\; 1\,, \qquad (F^{\mathcal{L}_1,\eta\mathcal{L}_1,\eta\mathcal{L}_3}_{\overline{\mathcal{L}_3}})_{\overline{\eta},\mathcal{L}_2} \;=\; \zeta^{-1}\,.$$

4.3 T-system

Definition 4.1 (T-system) *Let I be the set of invertible objects and N the set of non-invertible simple objects in the Haagerup-Izumi fusion ring with $G=\mathbb{Z}_{2n+1}$. The T-system is the set of constraints on the F-symbols*

$$\begin{aligned}
(F^{\eta,\mathcal{L}_2,\mathcal{L}_3}_{\mathcal{L}_4})_{\mathcal{L}_5,\mathcal{L}_6} \;&=\; (F^{\mathcal{L}_1,\eta,\mathcal{L}_3}_{\mathcal{L}_4})_{\mathcal{L}_5,\mathcal{L}_6} \;=\; (F^{\mathcal{L}_1,\mathcal{L}_2,\eta}_{\mathcal{L}_4})_{\mathcal{L}_5,\mathcal{L}_6} \;=\; (F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\eta}})_{\mathcal{L}_5,\mathcal{L}_6} \;=\; 1\,,\\
(F^{\eta\mathcal{L}\theta,\overline{\theta}\mathcal{L},\mathcal{L}}_{\eta\mathcal{L}})_{\eta,\overline{\theta}} \;&=\; (F^{\mathcal{L}_1,\eta\mathcal{L}_1,\eta\mathcal{L}_3}_{\mathcal{L}_3})_{\overline{\eta},\mathcal{L}_2} \;=\; \zeta^{-1}\,, \qquad (F^{\mathcal{L}_1,\mathcal{L}_3,\eta\mathcal{L}_3}_{\mathcal{L}_1\overline{\eta}})_{\mathcal{L}_2,\overline{\eta}} \;=\; 1\,,\\
(F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\mathcal{L}_4})_{\mathcal{L}_5,\mathcal{L}_6} \;&=\; (F^{\mathcal{L}_1\overline{\eta},\eta\mathcal{L}_2,\mathcal{L}_3}_{\mathcal{L}_4})_{\mathcal{L}_5,\eta\mathcal{L}_6} \;=\; (F^{\mathcal{L}_1,\mathcal{L}_2\overline{\eta},\eta\mathcal{L}_3}_{\mathcal{L}_4})_{\mathcal{L}_5\overline{\eta},\mathcal{L}_6}\\
&=\; (F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3\overline{\eta}}_{\mathcal{L}_4\overline{\eta}})_{\mathcal{L}_5,\mathcal{L}_6\overline{\eta}} \;=\; (F^{\eta\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\eta\mathcal{L}_4})_{\eta\mathcal{L}_5,\mathcal{L}_6}\,,
\end{aligned} \tag{4.1}$$

for all $\eta,\theta\in I$ and $\mathcal{L},\mathcal{L}_i\in N$.

[5]Generally, the values of loops in a nice gauge are solutions to the polynomial equations given by the abelianization of the fusion ring [11].

For the Haagerup-Izumi fusion ring with $G = \mathbb{Z}_{2n+1}$, the number of independent F-symbols after imposing the T-system constraints is $(2n+1)^2+1$. The F-symbols for any transparent Haagerup-Izumi category must satisfy the T-system constraints.

The number of independent F-symbols can be further reduced by the tetrahedral symmetry. Because all non-invertible simple objects are unoriented, every transparent Haagerup-Izumi category is spherical and the tetrahedron enjoys S_4 symmetry. Since the factors in the relations (2.3) and (2.4) between the tetrahedra and the F-symbols are universally equal to $\zeta^{-1}\xi^2$, the set of F-symbols with all non-invertible objects are invariant under the A_4 rotational symmetry of the tetrahedron, and complex conjugate under a reflection (with ξ chosen to be real). One may choose to be optimistic and postulate that the A_4 invariance is enhanced to S_4.

Table 1 lists the numbers of independent F-symbols after imposing the T-system constraints together with A_4 or S_4 tetrahedral invariance. With A_4 invariance (necessary consequence of transparency), the pentagon identity under the T-system constraints can be practically solved up to $G = \mathbb{Z}_9$ by computing a Groebner basis using MAGMA [39]. With S_4 invariance, it can be solved up to $G = \mathbb{Z}_{15}$.

G	A_4	S_4
$\mathbb{Z}_3$	8	7
$\mathbb{Z}_5$	22	16
$\mathbb{Z}_7$	44	29
$\mathbb{Z}_9$	74	46
$\mathbb{Z}_{11}$	112	67
$\mathbb{Z}_{13}$	158	92
$\mathbb{Z}_{15}$	212	121

Table 1: The numbers of independent F-symbols for the Haagerup-Izumi fusion rings after imposing the T-system constraints together with A_4 or S_4 tetrahedral invariance.

5 Classification of F-Symbols

5.1 Main theorems

Theorem 5.1 *For the Haagerup-Izumi fusion rings with $G = \mathbb{Z}_{2n+1}$, let*

$$\zeta_\pm \equiv \frac{2n+1 \pm \sqrt{(2n+1)^2+4}}{2}.$$

The pentagon identity under the T-system constraints (4.1) and A_4 tetrahedral invariance (necessary by transparency) has the following solutions:

(a) There are two solutions for $G = \mathbb{Z}_1$ corresponding to the Fibonacci and Lee-Yang categories.

(b) There are eight solutions for $G = \mathbb{Z}_3$.

(c) There are sixteen solutions for $G = \mathbb{Z}_5$.

(d) There are twenty-four solutions for $G = \mathbb{Z}_7$.

(e) For $G = \mathbb{Z}_{2n+1}$ with $n = 1, 2, 3$, the solutions form four order-$2n$ orbits of the $\mathbb{Z}_{2n}$ automorphism group. Two orbits are unitary with $\zeta = \zeta_+$; the F-symbols are real in one of the two orbits, and complex in the other. The remaining two orbits are the non-unitary Galois associates of the two unitary orbits, with $\zeta = \zeta_-$.

Theorem 5.2 *For the Haagerup-Izumi fusion rings with $G = \mathbb{Z}_{2n+1}$, let*

$$\zeta_\pm \equiv \frac{2n+1 \pm \sqrt{(2n+1)^2+4}}{2} \,.$$

The pentagon identity under the T-system constraints (4.1) and S_4 tetrahedral invariance has the following solutions:

(a) There are two solutions for $G = \mathbb{Z}_1$, corresponding to the Fibonacci and Lee-Yang categories.

(b) There are four solutions for $G = \mathbb{Z}_3$.

(c) There are eight solutions for $G = \mathbb{Z}_5$.

(d) There are twelve solutions for $G = \mathbb{Z}_7$.

(e) There are twenty-four solutions for $G = \mathbb{Z}_{13}$.

(f) For $G = \mathbb{Z}_{2n+1}$ with $n = 1, 2, 3, 6$, the solutions form two order-$2n$ orbits of the $\mathbb{Z}_{2n}$ automorphism group. One orbit is unitary with $\zeta = \zeta_+$, and the other orbit consists of the non-unitary Galois associates with $\zeta = \zeta_-$.

(g) There are twenty-four solutions for $G = \mathbb{Z}_9$, forming four order-six orbits of the $\mathbb{Z}_6$ automorphism group. Two orbits are unitary with $\zeta = \zeta_+$, and the other two orbits consist of the non-unitary Galois associates with $\zeta = \zeta_-$.

(h) *There are twenty-four solutions for $G = \mathbb{Z}_{11}$, forming two order-two orbits and two order-ten of the $\mathbb{Z}_{10}$ automorphism group. One order-two orbit and one order-ten orbit are unitary with $\zeta = \zeta_+$, and the other two orbits consist of the non-unitary Galois associates with $\zeta = \zeta_-$.*

(i) *There are forty-eight solutions for $G = \mathbb{Z}_{15}$, forming six order-eight orbits of the $\mathbb{Z}_2 \times \mathbb{Z}_4$ automorphism group. Three orbits are unitary with $\zeta = \zeta_+$, and the other three orbits consist of the non-unitary Galois associates with $\zeta = \zeta_-$.*

(j) *In the above, the F-symbols are real when $\zeta = \zeta_+$, and complex when $\zeta = \zeta_-$. Solutions in a single orbit of the automorphism group have the same $(F^{\rho,\rho,\rho}_{\rho})_{\rho,\rho}$, while different orbits have distinct $(F^{\rho,\rho,\rho}_{\rho})_{\rho,\rho}$. Since $(F^{\rho,\rho,\rho}_{\rho})_{\rho,\rho}$ is gauge-invariant, solutions with distinct values correspond to inequivalent fusion categories.*

Let I be the set of invertible objects and N be the set of non-invertible simple objects of the Haagerup-Izumi fusion ring with $G = \mathbb{Z}_{2n+1}$. By (4.1), the F-symbols with at least one invertible object are given by

$$(F^{\eta\mathcal{L}\theta,\overline{\mathcal{L}\theta},\mathcal{L}}_{\eta\mathcal{L}})_{\eta,\bar{\theta}} \;=\; (F^{\mathcal{L}_1,\eta\mathcal{L}_1,\eta\mathcal{L}_3}_{\overline{\mathcal{L}_3}})_{\bar{\eta},\mathcal{L}_2} \;=\; \zeta^{-1}\,, \qquad (F^{\mathcal{L}_1,\mathcal{L}_3,\eta\mathcal{L}_3}_{\overline{\eta\mathcal{L}_1}})_{\mathcal{L}_2,\bar{\eta}} \;=\; 1\,,$$

for all $\eta, \theta \in I$ and $\mathcal{L}_i \in N$. For the F-symbols with all simple objects non-invertible, it suffices to specify the $(2n+1)^2$ components $(F^{\rho,\rho,\rho}_{*})_{\rho,*}$ with $*$ running over the non-invertible simple objects. The rest are equal to one of the above by the $\mathbb{Z}^4_{2n+1}$ invariance relations (4.1). In fact, these invariance relations can be equivalently written as

$$\begin{aligned}(F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\mathcal{L}_4})_{\mathcal{L}_5,\mathcal{L}_6} \;&=\; (F^{\eta\mathcal{L}_1,\eta\mathcal{L}_2,\eta\mathcal{L}_3}_{\eta\mathcal{L}_4})_{\eta\mathcal{L}_5,\eta\mathcal{L}_6} \;=\; (F^{\eta\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3\eta}_{\mathcal{L}_4})_{\mathcal{L}_5,\mathcal{L}_6} \\ &=\; (F^{\mathcal{L}_1,\eta\mathcal{L}_2,\mathcal{L}_3}_{\mathcal{L}_4\eta})_{\mathcal{L}_5,\mathcal{L}_6} \;=\; (F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\mathcal{L}_4})_{\eta\mathcal{L}_5,\mathcal{L}_6\eta}\,,\end{aligned}$$

for all $\eta, \theta \in I$ and $\mathcal{L}_i \in N$. Note that the equality of the first and the last terms implies that every $F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\mathcal{L}_4}$ is an anti-circulant matrix.

Since the pentagon identity is a polynomial equation, the F-symbols are roots of polynomials. In the following, x_i (resp. other symbols) denotes the i-th (in increasing order) real root of some polynomial in x (resp. other symbols). This notation is unambiguous because there are no multiple roots. The simpler polynomials are given in the main text, while the more complicated ones are given in Appendix A.

5.2 Haagerup $\mathcal{H}_3$ ($G = \mathbb{Z}_3$)

Under the automorphism group $\mathrm{Aut}(G) \cong \mathbb{Z}_2$, there is exactly one unitary orbit with two solutions. One solution is given by

$F_*^{\rho,\rho,\rho}(\rho,*)$	ρ	$\alpha\rho$	$\alpha^2\rho$
ρ	x	y_1	y_2
$\alpha\rho$	y_1	y_2	z
$\alpha^2\rho$	y_2	z	y_1

where

$$x = \frac{2-\sqrt{13}}{3}, \quad y_{1,2} = \frac{1}{12}\left(5-\sqrt{13} \mp \sqrt{6\left(1+\sqrt{13}\right)}\right), \quad z = \frac{1+\sqrt{13}}{6}.$$

$\mathrm{Aut}(G) \cong \mathbb{Z}_2$ exchanges y_1 and y_2, giving the other solution in the same orbit.

5.3 $G = \mathbb{Z}_5$

Under the automorphism group $\mathrm{Aut}(G) \cong \mathbb{Z}_4$, there is exactly one unitary orbit with four solutions. One solution is given by

$F_*^{\rho,\rho,\rho}(\rho,*)$	ρ	$\alpha\rho$	$\alpha^2\rho$	$\alpha^3\rho$	$\alpha^4\rho$
ρ	x	y_1	y_3	y_2	y_4
$\alpha\rho$	y_1	y_4	z_2	z_4	z_2
$\alpha^2\rho$	y_3	z_2	y_2	z_4	z_4
$\alpha^3\rho$	y_2	z_4	z_4	y_3	z_2
$\alpha^4\rho$	y_4	z_2	z_4	z_2	y_1

where

$$x = \frac{7-\sqrt{29}}{5},$$

y_i are the real roots of

$$P_y^{\mathbb{Z}_5}(y) = 625y^8 - 1375y^7 + 1275y^6 + 245y^5 - 654y^4 + 152y^3 + 75y^2 - 29y - 1,$$

and z_i are the real roots of

$$P_z^{\mathbb{Z}_5}(z) = 25z^4 - 15z^3 - 9z^2 + 7z - 1.$$

$\mathrm{Aut}(G) \cong \mathbb{Z}_4$ permutes y_i and exchanges z_2 and z_4 by

$$\tau_y = (1243), \quad \tau_z = (24),$$

giving the other solutions in the same orbit.

Note that the polynomial in z factorizes over $\mathbb{Q}(\sqrt{29 = 5^2 + 4})$, and z_2, z_4 are the roots to one of the factors. This pattern continues in the following solutions. Namely, all polynomials factorize over $\mathbb{Q}(\sqrt{n^2 + 4})$, and the roots in the same orbit of $\mathrm{Aut}(G)$ will always be roots of the same factor.

5.4 $G = \mathbb{Z}_7$

Under the automorphism group $\mathrm{Aut}(G) \cong \mathbb{Z}_6$, there is exactly one unitary orbit with six solutions. One solution is given by

$F_*^{\rho,\rho,\rho}(\rho,*)$	ρ	$\alpha\rho$	$\alpha^2\rho$	$\alpha^3\rho$	$\alpha^4\rho$	$\alpha^5\rho$	$\alpha^6\rho$
ρ	x	y_1	y_2	y_6	y_4	y_3	y_5
$\alpha\rho$	y_1	y_5	z_6	w_2	z_3	w_1	z_6
$\alpha^2\rho$	y_2	z_6	y_3	w_1	z_4	z_4	w_2
$\alpha^3\rho$	y_6	w_2	w_1	y_4	z_3	z_4	z_3
$\alpha^4\rho$	y_4	z_3	z_4	z_3	y_6	w_2	w_1
$\alpha^5\rho$	y_3	w_1	z_4	z_4	w_2	y_2	z_6
$\alpha^6\rho$	y_5	z_6	w_2	z_3	w_1	z_6	y_1

where

$$x = \frac{11 - 2\sqrt{53}}{7} \,.$$

$\mathrm{Aut}(G) \cong \mathbb{Z}_6 \cong \langle \sigma, \tau \mid \sigma^2 = \tau^3 = 1 \rangle$ permutes the roots by

$$\begin{aligned} \sigma_y &= (15)(23)(46)\,, \quad \sigma_z = \mathrm{id}\,, \quad \sigma_w = (12)\,, \\ \tau_y &= (356)(142)\,, \quad \tau_z = (346)\,, \quad \tau_w = \mathrm{id}\,, \end{aligned}$$

giving the other solutions in the same orbit.

5.5 $G = \mathbb{Z}_9$

Under the automorphism group $\mathrm{Aut}(G) \cong \mathbb{Z}_6$, there are two unitary orbits each with six solutions. A solution in one orbit is given by

$F_*^{\rho,\rho,\rho}(\rho,*)$	ρ	$\alpha\rho$	$\alpha^2\rho$	$\alpha^3\rho$	$\alpha^4\rho$	$\alpha^5\rho$	$\alpha^6\rho$	$\alpha^7\rho$	$\alpha^8\rho$
ρ	x_1	y_1	y_{12}	$\textcircled{r}_4$	y_6	y_8	$\textcircled{r}_1$	y_7	y_5
$\alpha\rho$	y_1	y_5	z_8	w_{10}	w_2	z_{11}	w_5	w_7	z_8
$\alpha^2\rho$	y_{12}	z_8	y_7	w_7	z_4	w_9	w_1	z_4	w_{10}
$\alpha^3\rho$	$\textcircled{r}_4$	w_{10}	w_7	$\textcircled{r}_1$	w_5	w_1	$\textcircled{s}_4$	w_9	w_2
$\alpha^4\rho$	y_6	w_2	z_4	w_5	y_8	z_{11}	w_9	w_1	z_{11}
$\alpha^5\rho$	y_8	z_{11}	w_9	w_1	z_{11}	y_6	w_2	z_4	w_5
$\alpha^6\rho$	$\textcircled{r}_1$	w_5	w_1	$\textcircled{s}_4$	w_9	w_2	$\textcircled{r}_4$	w_{10}	w_7
$\alpha^7\rho$	y_7	w_7	z_4	w_9	w_1	z_4	w_{10}	y_{12}	z_8
$\alpha^8\rho$	y_5	z_8	w_{10}	w_2	z_{11}	w_5	w_7	z_8	y_1

where

$$x_{1,2} = \frac{35 - 4\sqrt{85} \mp \sqrt{517 - 56\sqrt{85}}}{18}$$

are the two negative roots of

$$P_x^{\mathbb{Z}_9}(x) = 81x^4 - 630x^3 + 899x^2 + 210x + 9\,.$$

$\mathrm{Aut}(G) \cong \mathbb{Z}_6 \cong \langle \sigma, \tau \mid \sigma^2 = \tau^3 = 1 \rangle$ permutes the roots by

$$\begin{aligned}
&\sigma_x = \mathrm{id}, \quad \sigma_y = (1\ 5)(2\ 4)(3\ 11)(6\ 8)(7\ 12)(9\ 10)\,, \quad \sigma_z = \mathrm{id}\,,\\
&\sigma_w = (1\ 9)(2\ 5)(3\ 8)(4\ 12)(6\ 11)(7\ 10)\,, \quad \sigma_r = (1\ 4)(2\ 3)\,, \quad \sigma_s = \mathrm{id}\,,\\
&\tau_x = \mathrm{id}, \quad \tau_y = (1\ 6\ 7)(2\ 3\ 9)(4\ 11\ 10)(5\ 8\ 12)\,, \quad \tau_z = (3\ 10\ 7)(4\ 8\ 11)\,,\\
&\tau_w = (1\ 7\ 2)(3\ 6\ 12)(4\ 8\ 11)(5\ 9\ 10)\,, \quad \tau_r = \mathrm{id}, \quad \tau_s = \mathrm{id}\,,
\end{aligned}$$

giving the other solutions in the same orbit. There is an additional $\mathbb{Z}_2 \cong \langle \iota \mid \iota^2 = 1 \rangle$ action that acts by

$$\begin{aligned}
&\iota_x = (1\ 2)\,, \quad \iota_y = (1\ 2)(3\ 6)(4\ 5)(7\ 9)(8\ 11)(10\ 12)\,, \quad \iota_z = (3\ 4)(7\ 11)(8\ 10)\,,\\
&\iota_w = (1\ 12)(2\ 6)(3\ 7)(4\ 9)(5\ 11)(8\ 10)\,, \quad \iota_r = (1\ 3)(2\ 4)\,, \quad \iota_s = (2\ 4)\,,
\end{aligned}$$

and exchanges the two unitary orbits.

5.6 $G = \mathbb{Z}_{11}$

Under the automorphism group $\mathrm{Aut}(G) \cong \mathbb{Z}_{10}$, there is one unitary orbit with two solutions and one unitary orbit with ten solutions. In the orbit with two solutions, one solution is given by

$F^{\rho,\rho,\rho}_*(\rho,*)$	ρ	$\alpha\rho$	$\alpha^2\rho$	$\alpha^3\rho$	$\alpha^4\rho$	$\alpha^5\rho$	$\alpha^6\rho$	$\alpha^7\rho$	$\alpha^8\rho$	$\alpha^9\rho$	$\alpha^{10}\rho$
ρ	x	y_1	y_2	y_1	y_1	y_1	y_2	y_2	y_2	y_1	y_2
$\alpha\rho$	y_1	y_2	z_2	w_2	w_2	w_1	z_2	w_2	w_1	w_1	z_2
$\alpha^2\rho$	y_2	z_2	y_1	w_1	z_2	w_2	w_1	w_2	w_1	z_2	w_2
$\alpha^3\rho$	y_1	w_2	w_1	y_2	w_1	w_1	z_2	z_2	z_2	w_2	w_2
$\alpha^4\rho$	y_1	w_2	z_2	w_1	y_2	w_2	w_2	z_2	z_2	w_1	w_1
$\alpha^5\rho$	y_1	w_1	w_2	w_1	w_2	y_2	z_2	w_1	z_2	w_2	z_1
$\alpha^6\rho$	y_2	z_2	w_1	z_2	w_2	z_2	y_1	w_1	w_2	w_1	w_2
$\alpha^7\rho$	y_2	w_2	w_2	z_2	z_2	w_1	w_1	y_1	w_2	z_2	w_1
$\alpha^8\rho$	y_2	w_1	w_1	z_2	z_2	z_2	w_2	w_2	y_1	w_2	w_1
$\alpha^9\rho$	y_1	w_1	z_2	w_2	w_1	w_2	w_1	z_2	w_2	y_2	z_2
$\alpha^{10}\rho$	y_2	z_2	w_2	w_2	w_1	z_1	w_2	w_1	w_1	z_2	y_1

where

$$x = \frac{13 - 5\sqrt{5}}{11}$$

is a root of the polynomial

$$P^{\mathbb{Z}_{11}}_{2|x}(x) = 11x^2 - 26x + 4\,.$$

The $\mathbb{Z}_2$ subgroup of $\mathrm{Aut}(G) \cong \mathbb{Z}_{10}$ exchanges y_1 with y_2 and w_1 with w_2. In the order-ten orbit, one solution is given by

$F^{\rho,\rho,\rho}_*(\rho,*)$	ρ	$\alpha\rho$	$\alpha^2\rho$	$\alpha^3\rho$	$\alpha^4\rho$	$\alpha^5\rho$	$\alpha^6\rho$	$\alpha^7\rho$	$\alpha^8\rho$	$\alpha^9\rho$	$\alpha^{10}\rho$
ρ	x	y_1	y_{10}	y_9	y_2	y_8	y_3	y_7	y_4	y_6	y_5
$\alpha\rho$	y_1	y_5	z_6	w_{10}	w_3	w_9	z_7	w_1	w_7	w_4	z_6
$\alpha^2\rho$	y_{10}	z_6	y_6	w_4	z_3	w_2	w_6	w_5	w_8	z_3	w_{10}
$\alpha^3\rho$	y_9	w_{10}	w_4	y_4	w_7	w_8	z_4	z_8	z_4	w_2	w_3
$\alpha^4\rho$	y_2	w_3	z_3	w_7	y_7	w_1	w_5	z_8	z_8	w_6	w_9
$\alpha^5\rho$	y_8	w_9	w_2	w_8	w_1	y_3	z_7	w_6	z_4	w_5	z_7
$\alpha^6\rho$	y_3	z_7	w_6	z_4	w_5	z_7	y_8	w_9	w_2	w_9	w_1
$\alpha^7\rho$	y_7	w_1	w_5	z_8	z_8	w_6	w_9	y_2	w_3	z_3	w_7
$\alpha^8\rho$	y_4	w_7	w_8	z_4	z_8	z_4	w_2	w_3	y_9	w_{10}	w_4
$\alpha^9\rho$	y_6	w_4	z_3	w_2	w_6	w_5	w_9	z_3	w_{10}	y_{10}	z_6
$\alpha^{10}\rho$	y_5	z_6	w_{10}	w_3	w_9	z_7	w_1	w_7	w_4	z_6	y_1

where

$$x = \frac{101 - 49\sqrt{5}}{22} x_{1,2} = \frac{101 \mp 49\sqrt{5}}{22}$$

is a root of the polynomial

$$P^{\mathbb{Z}_{11}}_{10|x}(x) = 11x^2 - 101x - 41.$$

$\mathrm{Aut}(G) \cong \mathbb{Z}_{10} \cong \langle \sigma, \tau \mid \sigma^2 = \tau^5 = 1 \rangle$ permutes the roots by

$$\sigma_y = (1\ 5)(2\ 7)(3\ 8)(4\ 9)(6\ 10)\,, \quad \sigma_z = \mathrm{id}\,, \quad \sigma_w = (1\ 9)(2\ 8)(3\ 7)(4\ 10)(5\ 6)\,,$$

$$\tau_y = (1\ 2\ 8\ 6\ 9)(3\ 10\ 4\ 5\ 7)\,, \quad \tau_z = (3\ 4\ 6\ 8\ 7)\,, \quad \tau_w = (1\ 5\ 2\ 10\ 3)(4\ 7\ 9\ 6\ 8)\,,$$

giving the other solutions in the same orbit.

5.7 $G = \mathbb{Z}_{13}$

Under the automorphism group $\mathrm{Aut}(G) \cong \mathbb{Z}_{12}$, there is exactly one unitary orbit with twelve solutions. One solution is given by

$F_*^{\rho,\rho,\rho}(\rho,*)$	ρ	$\alpha\rho$	$\alpha^2\rho$	$\alpha^3\rho$	$\alpha^4\rho$	$\alpha^5\rho$	$\alpha^6\rho$	$\alpha^7\rho$	$\alpha^8\rho$	$\alpha^9\rho$	$\alpha^{10}\rho$	$\alpha^{11}\rho$	$\alpha^{12}\rho$
ρ	x	y_1	y_9	y_{12}	y_8	y_4	y_7	y_3	y_5	y_2	y_{10}	y_6	y_{11}
$\alpha\rho$	y_1	y_{11}	z_6	w_5	s_3	w_8	w_4	z_9	w_{11}	w_9	s_2	w_2	z_6
$\alpha^2\rho$	y_9	z_6	y_6	w_2	z_7	w_{10}	w_{12}	s_1	s_4	w_7	w_1	z_7	w_5
$\alpha^3\rho$	y_{12}	w_5	w_2	y_{10}	s_2	w_1	z_{10}	w_3	z_8	w_6	z_{10}	w_{10}	s_3
$\alpha^4\rho$	y_8	s_3	z_7	s_2	y_2	w_9	w_7	w_6	z_4	z_4	w_3	w_{12}	w_8
$\alpha^5\rho$	y_4	w_8	w_{10}	w_1	w_9	y_5	w_{11}	s_4	z_8	z_4	z_8	s_1	w_4
$\alpha^6\rho$	y_7	w_4	w_{12}	z_{10}	w_7	w_{11}	y_3	z_9	s_1	w_3	w_6	s_4	z_9
$\alpha^7\rho$	y_3	z_9	s_1	w_3	w_6	s_4	z_9	y_7	w_4	w_{12}	z_{10}	w_7	w_{11}
$\alpha^8\rho$	y_5	w_{11}	s_4	z_8	z_4	z_8	s_1	w_4	y_4	w_8	w_{10}	w_1	w_9
$\alpha^9\rho$	y_2	w_9	w_7	w_6	z_4	z_4	w_3	w_{12}	w_8	y_8	s_3	z_7	s_2
$\alpha^{10}\rho$	y_{10}	s_2	w_1	z_{10}	w_3	z_8	w_6	z_{10}	w_{10}	s_3	y_{12}	w_5	w_2
$\alpha^{11}\rho$	y_6	w_2	z_7	w_{10}	w_{12}	s_1	s_4	w_7	w_1	z_7	w_5	y_9	z_6
$\alpha^{12}\rho$	y_{11}	z_6	w_5	s_3	w_8	w_4	z_9	w_{11}	w_9	s_2	w_2	z_6	y_1

where

$$x = \frac{107 - 8\sqrt{173}}{13}$$

is a root of the polynomial

$$P_x^{\mathbb{Z}_{13}}(x) = 13x^2 - 214x + 29\,.$$

$\mathrm{Aut}(G) \cong \mathbb{Z}_{12} \cong \langle \sigma, \tau \mid \sigma^4 = \tau^3 = 1 \rangle$ permutes the roots in the following way

$$\sigma_y = (1\ 4\ 11\ 5)(2\ 7\ 8\ 3)(6\ 12\ 9\ 10)\,, \quad \sigma_z = (4\ 9)(6\ 8)(7\ 10)\,,$$
$$\sigma_w = (1\ 5\ 10\ 2)(3\ 7\ 6\ 12)(4\ 9\ 11\ 8)\,, \quad \sigma_s = (1\ 3\ 4\ 2)\,,$$

$$\tau_y = (1\ 2\ 12)(3\ 6\ 5)(4\ 7\ 9)(8\ 10\ 11)\,, \quad \tau_z = (4\ 10\ 6)(7\ 8\ 9)\,,$$
$$\tau_w = (1\ 4\ 7)(2\ 8\ 3)(5\ 9\ 6)(10\ 11\ 12)\,, \quad \tau_s = \mathrm{id}\,.$$

5.8 $G = \mathbb{Z}_{15}$

Under the automorphism group $\mathrm{Aut}(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, there are three unitary orbits each with eight solutions. A solution in one orbit is given by[6]

$F_*^{\rho,\rho,\rho}(\rho,*)$	ρ	$\alpha\rho$	$\alpha^2\rho$	$\alpha^3\rho$	$\alpha^4\rho$	$\alpha^5\rho$	$\alpha^6\rho$	$\alpha^7\rho$	$\alpha^8\rho$	$\alpha^9\rho$	$\alpha^{10}\rho$	$\alpha^{11}\rho$	$\alpha^{12}\rho$	$\alpha^{13}\rho$	$\alpha^{14}\rho$
ρ	x_2	y_1	y_9	r_7	y_2	s_5	r_1	y_{23}	y_{16}	r_4	s_4	y_{17}	r_9	y_{19}	y_{18}
$\alpha\rho$	y_1	y_{18}	z_{14}	w_{14}	t_{10}	u_5	v_{19}	w_{13}	z_7	w_4	v_{23}	u_{10}	t_5	w_1	z_{14}
$\alpha^2\rho$	y_9	z_{14}	y_{19}	w_1	z_{15}	v_6	w_{10}	u_2	t_7	t_{12}	u_{12}	w_{20}	v_2	z_{15}	w_{14}
$\alpha^3\rho$	r_7	w_{14}	w_1	r_9	t_5	v_2	a_{11}	w_{18}	v_5	a_4	v_{10}	w_{19}	a_{11}	v_6	t_{10}
$\alpha^4\rho$	y_2	t_{10}	z_{15}	t_5	y_{17}	u_{10}	w_{20}	w_{19}	z_4	v_4	v_{13}	z_4	w_{18}	w_{10}	u_5
$\alpha^5\rho$	s_5	u_5	v_6	v_2	u_{10}	s_4	v_{23}	u_{12}	v_{10}	v_{13}	b_6	v_4	v_5	u_2	v_{19}
$\alpha^6\rho$	r_1	v_{19}	w_{10}	a_{11}	w_{20}	v_{23}	r_4	w_4	t_{12}	a_4	v_4	v_{13}	a_4	t_7	w_{13}
$\alpha^7\rho$	y_{23}	w_{13}	u_2	w_{18}	w_{19}	u_{12}	w_4	y_{16}	z_7	t_7	v_5	z_4	v_{10}	t_{12}	z_7
$\alpha^8\rho$	y_{16}	z_7	t_7	v_5	z_4	v_{10}	t_{12}	z_7	y_{23}	w_{13}	u_2	w_{18}	w_{19}	u_{12}	w_4
$\alpha^9\rho$	r_4	w_4	t_{12}	a_4	v_4	v_{13}	a_4	t_7	w_{13}	r_1	v_{19}	w_{10}	a_{11}	w_{20}	v_{23}
$\alpha^{10}\rho$	s_4	v_{23}	u_{12}	v_{10}	v_{13}	z_{22}	v_4	v_5	u_2	v_{19}	s_5	u_5	v_6	v_2	u_{10}
$\alpha^{11}\rho$	y_{17}	u_{10}	w_{20}	w_{19}	z_4	v_4	v_{13}	z_4	w_{18}	w_{10}	u_5	y_2	t_{10}	z_{15}	t_5
$\alpha^{12}\rho$	r_9	t_5	v_2	a_{11}	w_{18}	v_5	a_4	v_{10}	w_{19}	a_{11}	v_6	t_{10}	r_7	w_{14}	w_1
$\alpha^{13}\rho$	y_{19}	w_1	z_{15}	v_6	w_{10}	u_2	t_7	t_{12}	u_{12}	w_{20}	v_2	z_{15}	w_{14}	y_9	z_{14}
$\alpha^{14}\rho$	y_{18}	z_{14}	w_{14}	t_{10}	u_5	v_{19}	w_{13}	z_7	w_4	v_{23}	u_{10}	t_5	w_1	z_{14}	y_1

$\mathrm{Aut}(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \cong \langle \sigma, \tau \mid \sigma^2 = \tau^4 = 1 \rangle$ permutes the roots in the following way

$$\sigma_y = (1\ 18)(2\ 17)(9\ 19)(16\ 23)\,, \quad \sigma_r = (1\ 4)(7\ 9)\,, \quad \sigma_s = (4\ 5)\,, \quad \sigma_t = (5\ 10)(7\ 12)\,,$$
$$\sigma_r = (2\ 12)(5\ 10)\,, \quad \sigma_r = (1\ 4)(2\ 6)(4\ 13)(10\ 20)\,, \quad \sigma_w = (1\ 14)(4\ 13)(10\ 20)(18\ 19)\,,$$
$$\sigma_z = \mathrm{id}\,, \quad \sigma_a = \mathrm{id}\,, \quad \sigma_b = \mathrm{id}\,,$$

$$\tau_y = (1\ 19\ 2\ 23)(9\ 17\ 16\ 18)\,, \quad \tau_r = (1\ 7\ 4\ 9)\,, \quad \tau_s = \mathrm{id}\,, \quad \tau_t = (5\ 7)(10\ 12)\,,$$
$$\tau_u = (2\ 10\ 12\ 5)\,, \quad \tau_v = (2\ 13\ 5\ 23)(4\ 10\ 19\ 6)\,, \quad \tau_w = (1\ 10\ 19\ 13)(4\ 14\ 20\ 18)\,,$$
$$\tau_z = (4\ 7\ 14\ 15)\,, \quad \tau_a = (4\ 11)\,, \quad \tau_b = \mathrm{id}\,,$$

[6]The polynomials are rather long and thus omitted in writing.

giving the other solutions in the same orbit. There is an additional $\mathbb{Z}_3 \cong \langle \iota | \iota^3 = 1 \rangle$ action that acts by

$$
\begin{aligned}
&\iota_x = (1\ 2\ 3)\,, \quad \iota_y = (1\ 3\ 5)(2\ 10\ 14)(4\ 20\ 18)(6\ 11\ 17)(7\ 9\ 13)(8\ 12\ 19)(15\ 16\ 22)(21\ 24\ 23)\,,\\
&\iota_r = (1\ 6\ 8)(2\ 7\ 5)(3\ 4\ 10)(9\ 12\ 11)\,, \quad \iota_s = (1\ 5\ 3)(2\ 6\ 4)\,,\\
&\iota_t = (1\ 6\ 7)(2\ 5\ 8)(3\ 10\ 11)(4\ 12\ 9)\,, \quad \iota_u = (1\ 12\ 4)(2\ 6\ 7)(3\ 8\ 10)(5\ 11\ 9)\,,\\
&\iota_v = (1\ 23\ 15)(2\ 3\ 12)(4\ 18\ 16)(5\ 7\ 22)(6\ 20\ 8)(9\ 14\ 13)(10\ 17\ 21)(11\ 24\ 19)\,,\\
&\iota_w = (1\ 8\ 16)(2\ 18\ 11)(3\ 10\ 22)(4\ 9\ 6)(5\ 24\ 20)(7\ 21\ 13)(12\ 14\ 15)(17\ 19\ 23)\,,\\
&\iota_z = (4\ 5\ 11)(7\ 10\ 8)(14\ 19\ 17)(15\ 16\ 20)\,,\\
&\iota_a = (3\ 10\ 11)(4\ 8\ 6)\,, \quad \iota_b = (3\ 6\ 5)\,,
\end{aligned}
$$

and cycles through the three distinct unitary orbits. The polynomial for x is given by

$$3375x^6 - 116550x^5 + 620280x^4 - 926392x^3 + 41520x^2 + 88128x - 6912\,.$$

6 Conclusions and outlook

In this paper, the notion of a transparent fusion category is defined, and the F-symbols for transparent Haagerup-Izumi categories with $G = \mathbb{Z}_{2n+1}$ are classified up to $\mathbb{Z}_9$, and constructed up to $\mathbb{Z}_{15}$ by additionally assuming S_4 invariance. Various graph equivalences and F-symbol relations were derived from transparency, reducing the number of independent F-symbols from $\mathcal{O}(n^6)$ to $\mathcal{O}(n^2)$ and rendering the pentagon identity practically solvable. It would be interesting to construct transparent fusion categories for other fusion rings, such as other quadratic (or generalized near-group) fusion rings where the fusion of the invertible objects with a single non-invertible object generates all the non-invertible objects [40,26]. A promising family of fusion rings is the following. Introduce ν invertible objects

$$\mathcal{I}, \quad \alpha, \quad \alpha^2, \quad \cdots \quad \alpha^{\nu-1}$$

and $\nu+1$ non-invertible simple objects

$$\rho, \quad \alpha\rho, \quad \alpha^2\rho, \quad \cdots \quad \alpha^{\nu-1}\rho, \quad \mathcal{N}\,.$$

The fusion ring is

$$
\begin{aligned}
&\alpha^\nu = 1\,, \quad \alpha\rho = \rho\,\alpha^{\nu-1}\,, \quad \alpha\mathcal{N} = \mathcal{N}\alpha = \mathcal{N}\,,\\
&\rho^2 = \mathcal{I} + \mathcal{Z} + \mathcal{N}\,, \quad \mathcal{N}^2 = \mathcal{Y} + \mathcal{Z}\,, \quad \rho\mathcal{N} = \mathcal{N}\rho = \mathcal{Z} + \mathcal{N}\,,
\end{aligned}
$$

where

$$\mathcal{Y} \equiv \sum_{k=0}^{\nu-1} \alpha^k\,, \quad \mathcal{Z} \equiv \sum_{k=0}^{\nu-1} \alpha^k \rho\,.$$

When $\nu = 1$, this is none other than the $R_{\mathbb{C}}(\widehat{so}(3))_5$ fusion ring. This family of fusion rings generalize $R_{\mathbb{C}}(\widehat{so}(3))_5$, similar to how the Haagerup-Izumi fusion rings generalize Fibonacci.

Explicit F-symbols foster interesting applications. For instance, three-manifold invariants can be defined by F-symbols alone without the need of braiding [41,42]. In physics, one could study the gapped phase of (1+1)d quantum field theory with the Haagerup-Izumi categories by solving the topological quantum field theory, as was done in [11] for fusion categories of smaller rank. One could also study the crossing symmetry of defect operator four-point functions, and obtain universal bounds on the spectrum with the conformal bootstrap [43].

Acknowledgements

We are indebted to Yuji Tachikawa for bringing our attention to the Haagerup-Izumi categories, and to Matthew Titsworth for sharing the explicit F-symbols for the Haagerup $\mathcal{H}_3$ fusion categories. We thank Chi-Ming Chang, Terry Gannon, Tobias Osborne, Yuji Tachikawa and Yifan Wang for helpful comments and suggestions on the draft. This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of High Energy Physics, under Award Number DE-SC0011632. YL is supported by the Sherman Fairchild Foundation.

A Polynomials with F-symbols as roots

A.1 $G = \mathbb{Z}_7$

$$\begin{aligned}
P_y^{\mathbb{Z}_7}(y) &= 117649y^{12} - 453789y^{11} + 1145277y^{10} - 1070503y^9 + 882588y^8 - 284732y^7 \\
&\quad - 89977y^6 + 31488y^5 - 1828y^4 - 849y^3 + 381y^2 + 45y - 1\,, \\
P_z^{\mathbb{Z}_7}(z) &= 343z^6 + 196z^5 - 371z^4 + 27z^3 + 56z^2 - 9z - 1\,, \\
P_w^{\mathbb{Z}_7}(w) &= 49w^4 - 63w^3 + 15w^2 + 10w - 4\,.
\end{aligned}$$

A.2 $G = \mathbb{Z}_9$

$$
\begin{aligned}
P_y^{\mathbb{Z}_9}(y) = {} & 282429536481y^{24} - 2541865828329y^{23} + 13891349053584y^{22} - 42375665666331y^{21} \\
& + 93048845085738y^{20} - 163017616751046y^{19} + 191382870385035y^{18} \\
& - 91749046865085y^{17} - 71565147070767y^{16} + 121393466114850y^{15} \\
& - 42556511453652y^{14} - 23330326470255y^{13} + 20787803433577y^{12} \\
& - 1805958554210y^{11} - 2533403044422y^{10} + 632950992624y^{9} \\
& + 91558817982y^{8} - 30315392921y^{7} - 4655443748y^{6} + 986603649y^{5} \\
& + 182920180y^{4} - 28268573y^{3} - 1118977y^{2} - 127236y - 1801\,, \\
P_z^{\mathbb{Z}_9}(z) = {} & 531441z^{12} + 885735z^{11} - 1535274z^{10} - 121014z^{9} + 647352z^{8} - 79407z^{7} \\
& - 92863z^{6} + 18139z^{5} + 4928z^{4} - 1208z^{3} - 64z + 25z - 1\,, \\
P_w^{\mathbb{Z}_9}(w) = {} & 282429536481w^{24} - 1129718145924w^{23} + 1997927461773w^{22} - 1984755165147w^{21} \\
& + 1330918519878w^{20} - 791614850283w^{19} + 459695402118w^{18} - 222483700269w^{17} \\
& + 99182263023w^{16} - 47943836820w^{15} + 17026501158w^{14} - 3348784053w^{13} \\
& + 1374949378w^{12} - 621445880w^{11} - 329500476w^{10} + 412571852w^{9} - 148134014w^{8} \\
& + 18260969w^{7} + 2110023w^{6} - 806198w^{5} + 47683w^{4} + 6215w^{3} - 711w^{2} + 4w + 1\,. \\
P_r^{\mathbb{Z}_9}(r) = {} & 6561r^{8} - 8019r^{7} + 1377r^{6} - 792r^{5} + 3349r^{4} + 4r^{3} - 662r^{2} + 52r + 19\,, \\
P_s^{\mathbb{Z}_9}(s) = {} & 81s^{4} + 99s^{3} + 17s^{2} - 14s - 4\,.
\end{aligned}
$$

A.3 $G = \mathbb{Z}_{11}$

For the unitary orbit with two solutions

$$
\begin{aligned}
P^{\mathbb{Z}_{11}}_{2|y}(y) &= 121y^4 - 209y^3 + 82y^2 + 24y - 9\,,\\
P^{\mathbb{Z}_{11}}_{2|z}(z) &= 11z^2 + 7z + 1\,,\\
P^{\mathbb{Z}_{11}}_{2|w}(w) &= 121w^4 - 88w^3 + 38w^2 - 13w + 1\,.
\end{aligned}
$$

For the unitary orbit with ten solutions,

$$
\begin{aligned}
P^{\mathbb{Z}_{11}}_{10|y}(y) = {}& 25937424601y^{20} - 47158953820y^{19} + 1064291844165y^{18} + 4808654315960y^{17}\\
&+ 35564388240370y^{16} + 114903432126461y^{15} + 194232171940290y^{14}\\
&+ 126582540515475y^{13} - 21851286302395y^{12} - 65093840585730y^{11}\\
&- 20230205549333y^{10} + 6813959963720y^{9} + 4785911566905y^{8} + 360322446200y^{7}\\
&- 303249779065y^{6} - 76228721396y^{5} - 379548930y^{4} + 2142467760y^{3}\\
&+ 324308000y^{2} + 19299130y + 40207\,,\\
P^{\mathbb{Z}_{11}}_{10|z}(z) = {}& 161051z^{10} + 658845z^{9} - 971630z^{8} - 542080z^{7} + 322135z^{6}\\
&+ 105612z^{5} - 39815z^{4} - 6570z^{3} + 1960z^{2} + 70z - 19\,,\\
P^{\mathbb{Z}_{11}}_{10|w}(w) = {}& 25937424601w^{20} - 176846076825w^{19} + 592702305965w^{18} - 1134445659765w^{17}\\
&+ 1534818445765w^{16} - 1765089648718w^{15} + 1769544129045w^{14}\\
&- 1394768735745w^{13} + 776013578560w^{12} - 263088585485w^{11} + 20179458718w^{10}\\
&+ 32370728245w^{9} - 20820136235w^{8} + 6982550700w^{7} - 1450721110w^{6}\\
&+ 175316847w^{5} - 7539540w^{4} - 877925w^{3} + 133550w^{2} - 5960w + 71\,.
\end{aligned}
$$

A.4 $G = \mathbb{Z}_{13}$

$$
\begin{aligned}
P_y^{\mathbb{Z}_{13}}(y) = {} & 23298085122481y^{24} + 80647217731665y^{23} + 3069557179509834y^{22} \\
& + 41919543603471508y^{21} + 536909384312855190y^{20} + 4259352400707950897y^{19} \\
& + 19179161744641728596y^{18} + 47561155144008593243y^{17} + 63626358551986353149y^{16} \\
& + 40207662041712799114y^{15} + 1257635216859228766y^{14} - 13522223195096193305y^{13} \\
& - 6598116247933199625y^{12} + 1284137113065113340y^{11} + 938990747292838888y^{10} \\
& + 202797783582401196y^{9} - 32756778784407789y^{8} - 16526752437401584y^{7} \\
& - 933201395423678y^{6} + 349378912529867y^{5} + 53761577382743y^{4} + 1555890743172y^{3} \\
& - 87453542726y^{2} - 2773486466y + 28678361\,, \\
P_z^{\mathbb{Z}_{13}}(z) = {} & 4826809z^{12} + 34901542z^{11} - 124183228z^{10} - 57416398z^{9} + 51122838z^{8} + 3476850z^{7} \\
& - 4988283z^{6} + 418090z^{5} + 93250z^{4} - 14139z^{3} + 205z^{2} + 38z - 1\,, \\
P_w^{\mathbb{Z}_{13}}(w) = {} & 23298085122481w^{24} - 268824059105550w^{23} + 1610738618763716w^{22} \\
& - 4730805028787149w^{21} + 8265875258850053w^{20} - 9798763675027379w^{19} \\
& + 8948312751528579w^{18} - 6464842564613641w^{17} + 3087209293878385w^{16} \\
& - 284952516401007w^{15} - 771813881083466w^{14} + 531872957583864w^{13} \\
& - 107361616574952w^{12} - 39739582655570w^{11} + 27485052167132w^{10} \\
& - 4323332693485w^{9} - 1159653323459w^{8} + 583780092624w^{7} - 51758752951w^{6} \\
& - 19939454943w^{5} + 4746063302w^{4} + 131285807w^{3} - 111025779w^{2} + 2170222w \\
& + 898159\,, \\
P_s^{\mathbb{Z}_{13}}(s) = {} & 28561s^{8} - 24167s^{7} + 163930s^{6} - 225693s^{5} + 119817s^{4} - 26999s^{3} + 1045s^{2} + 546s \\
& - 67\,.
\end{aligned}
$$

References

[1] V. F. R. Jones, *Index for subfactors*, *Inventiones Mathematicae* **72** (Feb, 1983) 1–25.

[2] V. Jones and V. S. Sunder, *Introduction to subfactors*, vol. 234. Cambridge University Press, 1997.

[3] P. Etingof, D. Nikshych, and V. Ostrik, *On fusion categories*, *Annals of Mathematics* (2005) 581–642.

[4] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*, vol. 205. American Mathematical Soc., 2016.

[5] G. Moore and N. Seiberg, *Polynomial equations for rational conformal field theories*, *Physics Letters B* **212** (1988), no. 4 451–460.

[6] G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, *Communications in Mathematical Physics* **123** (1989), no. 2 177–254.

[7] E. Witten, *Quantum field theory and the Jones polynomial*, *Communications in Mathematical Physics* **121** (1989), no. 3 351–399.

[8] V. F. Jones, *von Neumann algebras in mathematics and physics*. American Mathematical Society, 1990.

[9] L. Bhardwaj and Y. Tachikawa, *On finite symmetries and their gauging in two dimensions*, *Journal of High Energy Physics* **2018** (2018), no. 3 189.

[10] Y. Tachikawa, *On gauging finite subgroups*, *SciPost Phys.* **8** (2020) 15.

[11] C.-M. Chang, Y.-H. Lin, S.-H. Shao, Y. Wang, and X. Yin, *Topological defect lines and renormalization group flows in two dimensions*, *Journal of High Energy Physics* **2019** (Jan, 2019).

[12] C. Itzykson, A. Jaffe, H. Lehmann, P. Mitter, I. Singer, and R. Stora, *Recent developments in gauge theories*. Springer, 1980.

[13] A. Kapustin and R. Thorngren, *Anomalies of discrete symmetries in various dimensions and group cohomology*, *arXiv preprint arXiv:1404.3230* (2014).

[14] R. Thorngren and Y. Wang, *Fusion Category Symmetry I: Anomaly In-Flow and Gapped Phases*, *arXiv preprint arXiv:1912.02817* (2019).

[15] A. Wassermann, *Quantum subgroups and vertex algebras*, *Lectures given at MSRI in Dec* (2000).

[16] M. Müger, *From subfactors to categories and topology I: Frobenius algebras in and Morita equivalence of tensor categories*, *Journal of Pure and Applied Algebra* **180** (2003), no. 1-2 81–157.

[17] A. Ocneanu, *Quantized groups, string algebras and Galois theory for algebras*, *Operator algebras and applications* **2** (1988) 119–172.

[18] S. Popa et al., *Classification of amenable subfactors of type II*, *Acta Mathematica* **172** (1994), no. 2 163–255.

[19] U. Haagerup, *Principal graphs of subfactors in the index range* $4 < [M:N] < 3+\sqrt{2}$, *Subfactors (Kyuzeso, 1993)* (1994) 1–38.

[20] M. Asaeda and U. Haagerup, *Exotic subfactors of finite depth with Jones indices* $(5+\sqrt{13})/2$ *and* $(5+\sqrt{17})/2$, *Communications in mathematical physics* **202** (1999), no. 1 1–63.

[21] M. Izumi, *The structure of sectors associated with Longo–Rehren inclusions II: examples*, *Reviews in Mathematical Physics* **13** (2001), no. 05 603–674.

[22] D. E. Evans and T. Gannon, *The Exoticness and Realisability of Twisted Haagerup–Izumi Modular Data*, *Communications in Mathematical Physics* **307** (Sep, 2011) 463–512.

[23] P. Grossman, M. Izumi, and N. Snyder, *The Asaeda–Haagerup fusion categories*, *Journal für die reine und angewandte Mathematik (Crelles Journal)* **2018** (2018), no. 743 261–305.

[24] P. Grossman and M. Izumi, *Drinfeld centers of fusion categories arising from generalized Haagerup subfactors*, *arXiv preprint arXiv:1501.07679* (2015).

[25] M. Izumi, *The classification of* 3^n *subfactors and related fusion categories*, *Quantum Topology* **9** (Jul, 2018) 473–562.

[26] P. Grossman and M. Izumi, *Infinite families of potential modular data related to quadratic categories*, *arXiv preprint arXiv:1906.07397* (2019).

[27] V. G. Turaev and O. Y. Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols*, *Topology* **31** (1992), no. 4 865–902.

[28] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, vol. 18. Walter de Gruyter GmbH & Co KG, 2020.

[29] M. A. Levin and X.-G. Wen, *String-net condensation: A physical mechanism for topological phases*, *Physical Review B* **71** (2005), no. 4 045110.

[30] G. W. Moore and G. Segal, *D-branes and K-theory in 2D topological field theory*, *arXiv preprint hep-th/0609042* (2006).

[31] A. Davydov, L. Kong, and I. Runkel, *Field theories with defects and the centre functor*, *Mathematical foundations of quantum field theory and perturbative string theory* **83** (2011) 71–128.

[32] M. Izumi, *Subalgebras of infinite C∗-algebras with finite Watatani indices I. Cuntz algebras*, *Communications in mathematical physics* **155** (1993), no. 1 157–182.

[33] D. E. Evans and T. Gannon, *Non-unitary fusion categories and their doubles via endomorphisms*, *Advances in Mathematics* **310** (Apr, 2017) 1–43.

[34] M. Titsworth. Private communication: to appear.

[35] T. J. Osborne, D. E. Stiegemann, and R. Wolf, *The F-Symbols for the $\mathcal{H}_3$ Fusion Category*, `arXiv:1906.01322`.

[36] R. Penrose, *Applications of negative dimensional tensors*, *Combinatorial mathematics and its applications* **1** (1971) 221–244.

[37] J. W. Barrett and B. W. Westbury, *Spherical Categories*, *Advances in Mathematics* **143** (May, 1999) 357–375.

[38] P. Bruillard, *Rank 4 premodular categories*, *New York J. Math* **22** (2016) 775–800.

[39] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, *J. Symbolic Comput.* **24** (1997), no. 3-4 235–265. Computational algebra and number theory (London, 1993).

[40] J. Thornton, *Generalized near-group categories*, .

[41] J. Barrett and B. Westbury, *Invariants of piecewise-linear 3-manifolds*, *Transactions of the American Mathematical Society* **348** (1996), no. 10 3997–4022.

[42] S. Gelfand and D. Kazhdan, *Invariants of three-dimensional manifolds*, *Geometric & Functional Analysis GAFA* **6** (1996), no. 2 268–300.

[43] D. Simmons-Duffin, *The conformal bootstrap*, in *New Frontiers in Fields and Strings: TASI 2015 Proceedings of the 2015 Theoretical Advanced Study Institute in Elementary Particle Physics*, pp. 1–74. World Scientific, 2017.